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Gauge Symmetries and Dirac Conjecture

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Abstract The gauge symmetries of a constrained system can be deduced from the gauge identities with Lagrange method, or the first-class constraints with Hamilton approach. If Dirac conjecture is valid to a dynamic system, in which all the first-class constraints are the generators of the gauge transformations, the gauge transformations deduced from the gauge identities are consistent with these given by the first-class constraints. Once the equivalence vanishes to a constrained system, in which Dirac conjecture would be invalid. By using the equivalence, two counterexamples and one example to Dirac conjecture are discussed to obtain defined results.

Keywords Constrained Hamiltonian system · Gauge symmetry · Gauge identities · Dirac conjecture

1 Introduction

Dirac theory of constrained systems plays an important role in modern quantum field theory. By using it, many of the central problems, which appeared in the development of the quan-

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tization procedure of the gauge fields, have been solved. However, in spite of these general achievements some basic problems in this theory are still widely discussed. One of them is Dirac conjecture. Dirac in his work on the generalized canonical formalism conjectured that all first-class constraints (FCC) are independent generators of the gauge transformations, which generate equivalence transformations among physical states [1, 2]. In general, it is closely connected with the question of the possible equivalence between Dirac procedure in terms of the extended Hamiltonian H_E (containing the canonical Hamiltonian H_C and all FCCs and second-class constraints (SCC)) and the Lagrangian description. If Dirac conjecture holds true, the dynamics of a system possessing mixed constraints (including FCCs and SCCs) should be described by the motion equations deduced from the extended Hamiltonian H_E, and its conservation laws deduced from H_E through the canonical Noether theorem should be equivalent to the results arising from the Lagrangian formalism via the classical Noether theorem. Inversely, if the equivalence holds true to a constrained system, Dirac conjecture would be valid. By using the equivalence, some counterexamples to Dirac conjecture have been discussed [3–7].

In this paper this problem will be discussed from another new point of view. Basing on the gauge symmetries of the system, let us consider whether the gauge transformations deduced from the generators via Hamiltonian formalism are equivalent to these arising from the gauge identities through Lagrangian formalism. In the former case, these gauge transformations are generated, according to Dirac conjecture, by all FCCs. In the latter method the gauge transformations are reflected in the existence of the gauge identities [8–14]. Lagrange approach has been used to discuss constrained systems only containing FCCs [12] or with mixed constraints [13]. In our paper, the validity of Dirac conjecture to a constrained system is determined by the equivalence between the gauge transformations deduced from all FCCs in Hamiltonian formalism and these derived from the gauge identities in Lagrangian framework. By using it, we will discuss two counterexamples and an example to Dirac conjecture. The former two supply the result that the gauge transformations via Lagrangian method is not consistent with these through Hamiltonian approach. The later one has the gauge transformations which in Lagrangian and Hamiltonian formalisms are uniform. These results are completely consistent with the validity of Dirac conjecture.

This paper is organized as follows. In Sect. 2, the deducing process of the gauge transformations is reviewed in Lagrangian formalism. Section 3 supplies two counterexamples to the Dirac conjecture. Section 4 supplies an example to Dirac conjecture. Section 5 gives the conclusion.

2 Dynamics of System with Singular Lagrangian

For the sake of simplicity, a system with a Lagrangian $L(q^i, \dot{q}^i)$ is considered in the k-dimensional space. Based on Lagrangian equation, the motion equations for this system are

$$L_i \equiv W_{ij}\ddot{q}^j + \alpha_i = 0 \tag{1}$$

where L_i is the ith element of the primary motion equation matrix L, W_{ij} denotes an element of the Hess matrix W, the subindexes i and j denote the ith row and the jth column, as

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}, \tag{2}$$

and α_i is defined as

$$\alpha_i = \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j - \frac{\partial L}{\partial q^i}. \tag{3}$$

For a dynamic system with singular Lagrangian, the determinant of the Hess matrix is zero, directly speaking, (1) can not supply analytical solutions to all of the generalized accelerations $\ddot{q}^i$. If the rank of the Hess matrix is assumed as $(k - A_1)$ (k is the full rank of the Hess matrix), the Hess matrix has A_1 zero eigenvectors as λ^{a_1}, which satisfy

$$\lambda_i^{a_1} W_{ij} = 0 \quad (a_1 = 1, 2, \ldots, A_1). \tag{4}$$

Take the eigenvectors λ^{a_1} to left multiply (1), and A_1 equations obtain

$$\gamma^{a_1}(q, \dot{q}) = \lambda_i^{a_1} L_i = \lambda_i^{a_1}\alpha_i = 0 \quad (a_1 = 1, 2, \ldots, A_1) \tag{5}$$

where γ^{a_1} called gauge identities denote the functions depending the generalized coordinate q and the generalized velocity $\dot{q}$. If all the identities are not independent from each other, (5) would not supply analytical solutions to all the rest undefined accelerations in (1). And if we assume the rank of the matrix corresponding to (5) is $\bar{A}_1$, (5) contains $\bar{A}_1$ independent constraints, which are called primary gauge identities, greatly corresponding to the primary constraints in Hamiltonian formalism,

$$\gamma^{\bar{a}_1}(q, \dot{q}) = \lambda_i^{\bar{a}_1} L_i = 0 \quad (\bar{a}_1 = 1, 2, \ldots, \bar{A}_1). \tag{6}$$

In A_1 gauge identities, there are $\bar{A}_1$ ones independent, the rest $\hat{A}_1$ ones can be expanded as the sum of the $\bar{A}_1$ primary gauge identities as

$$\lambda^{\hat{a}_1}(q, \dot{q}) = \sum_{\bar{a}_1}^{\bar{A}_1} C_{\bar{a}_1}^{\hat{a}_1}(q, \dot{q})\lambda^{\bar{a}_1}(q, \dot{q}) \quad (\hat{a}_1 = 1, 2, \ldots, \hat{A}_1 = A_1 - \bar{A}_1), \tag{7}$$

where $C_{a_1}^{\hat{a}_1}$ is combinatorial coefficient. And Euler differential identities can be given by

$$\lambda_i^{\hat{a}_1} L_i = 0 \quad (\hat{a}_1 = 1, 2, \ldots, \hat{A}_1). \tag{8}$$

Addition of the total time differential of (6) expands the motion equations (1) as

$$\begin{cases} W_{ij}\ddot{q}^j + \alpha_i = 0 & (i = 1, 2, \ldots, k), \\ \frac{d\gamma^{\bar{a}_1}}{dt} = \frac{\partial \gamma^{\bar{a}_1}}{\partial \dot{q}^j}\ddot{q}^j + \frac{\partial \gamma^{\bar{a}_1}}{\partial q^j}\dot{q}^j = 0 & (\bar{a}_1 = 1, 2, \ldots, \bar{A}_1). \end{cases} \tag{9}$$

Integrate the second equation into the first, and the motion equations (9) are taken by

$$L_{i_1}^1 \equiv W_{i_1 j}^1 \ddot{q}^j + \alpha_{i_1}^1 = 0 \quad (i_1 = 1, 2, \ldots, k + \bar{A}_1), \tag{10}$$

where $L_{i_1}^1$ is the i_1th element of the second-order motion equations matrix L^1, $W_{i_1 j}^1$ denotes the i_1th row and the jth column element of the second-order Hess matrix W^1 including W_{ij} and $\frac{\partial \gamma^{\bar{a}_1}}{\partial \dot{q}^j}$, and α_{i_1} contains α_i and $\frac{\partial \gamma^{\bar{a}_1}}{\partial q^j}\dot{q}^j$.

Iterate the steps of (4)–(8), and one obtains

$$\lambda_i^{\hat{a}_2} L_i + \lambda_i^{\hat{a}_2}\frac{d}{dt}(\lambda_i^{\bar{a}_1} L_i) = 0 \quad (\hat{a}_2 = 1, 2, \ldots, \hat{A}_2), \tag{11}$$

here $\hat{A}_2$ is the number of the secondary gauge identities, and $\lambda_i^{\hat{a}_2}$ denotes independent nonzero eigenvector.

Following the previously iterating process, the nth fold obtains the motion equations of the generalized accelerations as

$$W^n_{i_n j}\ddot{q}^j + \alpha_{i_n} = 0 \quad (i_n = 1, 2, \ldots, k + \bar{A}_1 + \bar{A}_2 + \cdots + \bar{A}_n). \tag{12}$$

In the nth step, the addition of $\bar{A}_n$ new motion equations about generalized accelerations expands the original motion equations (1) again. To the matrix W^n, there are A_n new eigenvectors, they contain $\bar{A}_{n+1}$ new gauge identities, and add $\hat{A}_{n+1}$ differential equations to the Euler relation expressions as

$$\lambda_i^{\hat{a_{n+1}}} L_i + \lambda_i^{\hat{a_{n+1}}} \frac{d\gamma^{\bar{a}_1}}{dt} + \cdots + \lambda_i^{\hat{a_{n+1}}} \frac{d\gamma^{\bar{a}_n}}{dt} = 0. \tag{13}$$

Take the analogy of (11), (13) is converted, by partially differential, into

$$\sum_s^n \frac{d^s}{dt^s}(\phi_s^i L_i) \equiv 0, \tag{14}$$

where ϕ_s^i is a function depending on the generalized coordinates, the generalized velocities, all the zero eigenvectors and their differentials, and is the associated coefficient relating the above identity to the corresponding matrix element L_i. Based on (14), the gauge transformations deduced from the gauge identities in Lagrangian formalism can be expressed as

$$\delta q^i = \sum_{k,s} (-1)^k \frac{d^k w^s(t)}{dt^k} \phi_s^i(q, \dot{q}), \tag{15}$$

where $w^s(t)$ is the gauge parameter, an infinitesimal time dependent function.

3 Two Counterexamples to Dirac Conjecture

A counterexample to Dirac conjecture is considered with Lagrangian [15]

$$L = \frac{1}{2}(e^{2u(y)}\dot{x}^2 + e^{-2v(-y)}\dot{z}^2), \tag{16}$$

where $u(y)$ and $v(-y)$ satisfy the following equations

$$u''(y) = u'(y) + 2[u'(y)]^2, \tag{17}$$

and

$$-v''(-y) = v'(-y) + 2[v'(-y)]^2. \tag{18}$$

With Dirac-Bergamman method, and in Hamiltonian formalism [11, 15–20], this dynamic system is discussed. With respect to the canonical variables x, y, z, the canonical momenta p_x, p_y, p_z are

$$p_x = \frac{\partial L}{\partial \dot{x}} = e^{2u(y)}\dot{x}, \qquad p_y = 0, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = e^{-2v(-y)}\dot{z}, \tag{19}$$

respectively. In Dirac-Bergamman framework, all constraints (including primary constraints and secondary) are easily deduced as

$$\phi_0 = p_y \approx 0, \tag{20}$$

and

$$\phi_1 = e^{-2u(y)} p_x^2 u'(y) + e^{2v(-y)} p_z^2 v'(-y) \approx 0. \tag{21}$$

And it is easy to see that the two constraints (20) and (21) are FCCs. The generator of the gauge transformations of the system is, therefore,

$$G = \varepsilon_0(t)\phi_0 - \dot{\varepsilon}_0(t)\phi_1, \tag{22}$$

which can give rise to gauge transformations

$$\begin{cases} \delta x = [x, G] = -2\dot{\varepsilon}_0(t) e^{-2u(y)} p_x u'(y), \\ \delta y = [y, G] = \varepsilon_0(t), \\ \delta z = [z, G] = -2\dot{\varepsilon}_0(t) e^{2v(-y)} p_z v'(-y), \\ \delta p_x = [p_x, G] = 0, \\ \delta p_y = [p_y, G] = -\dot{\varepsilon}_0(t)\phi_1, \\ \delta p_z = [p_z, G] = 0. \end{cases} \tag{23}$$

In what follows, the gauge transformations are deduced in Lagrangian formalism. Based on (16) and (1), the motion equation of the system (16) can be written as

$$L_i^0 \equiv W_{ij}^0 \ddot{q}^j + \alpha_i^0 = 0 \quad (i, j = 1, 2, 3), \tag{24}$$

where

$$q = (q_i) = (q_1, q_2, q_3) = (x, y, z), \tag{25}$$

and

$$W^0 = \begin{pmatrix} e^{2u(y)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-2v(-y)} \end{pmatrix}, \tag{26}$$

and

$$\alpha^0 = \begin{pmatrix} 2\dot{x}\dot{y}e^{2u(y)}u'(y) \\ -\dot{x}^2 e^{2u(y)} u'(y) - \dot{z}^2 e^{-2v(-y)} v'(-y) \\ 2\dot{y}\dot{z}e^{-2v(-y)} v'(-y) \end{pmatrix}. \tag{27}$$

To the matrix W^0, a zero-order zero eigenvector λ^0 can be given by

$$\lambda^0 = (0, 1, 0), \tag{28}$$

which satisfies the following equation

$$\lambda^0 W^0 = 0. \tag{29}$$

By left-multiplying the zero eigenvector to (24), a primary gauge identity appears

$$\gamma^0 = \lambda_i^0 L_i^0 = \lambda_i^0 \alpha_i^0 = -\dot{x}^2 e^{2u(y)} u'(y) - \dot{z}^2 e^{-2v(-y)} v'(-y) = 0, \tag{30}$$

where subindex i denotes the ith element of the matrix, and its total time differential is

$$\begin{aligned}\frac{d\gamma^0}{dt} &= -2\dot{x}e^{2u(y)}u'(y)\ddot{x} - 2\dot{z}e^{-2v(-y)}v'(-y)\ddot{z} \\ &\quad - \dot{x}^2\dot{y}e^{2u(y)}(2u'^2(y) + u''(y)) - \dot{y}\dot{z}^2e^{-2v(-y)}(2v'^2(-y) - v''(-y)) \\ &= 0,\end{aligned} \tag{31}$$

Substitute (31) into (24), and the motion equation (24) is taken by

$$\begin{cases} W^0_{ij}\ddot{q}^j + \alpha^0_i = 0, \\ \frac{d\gamma^0}{dt} = 0. \end{cases} \tag{32}$$

The first equation of (32) contains three Euler-Lagrangian equations for L_1, L_2, and L_3. Take the second equation of (32) as an additive motion equation L_4, then (32) can be taken by

$$L^1_{i_1} \equiv W^1_{i_1 j}\ddot{q}^j + \alpha^1_{i_1} = 0 \quad (j = 1, 2, 3;\ i_1 = 1, 2, 3, 4), \tag{33}$$

where

$$W^1 = \begin{pmatrix} e^{2u(y)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-2v(-y)} \\ -2\dot{x}e^{2u(y)}u'(y) & 0 & -2\dot{z}e^{-2v(-y)}v'(-y) \end{pmatrix}, \tag{34}$$

and

$$\alpha^1 = \begin{pmatrix} 2\dot{x}\dot{y}e^{2u(y)}u'(y) \\ -\dot{x}^2e^{2u(y)}u'(y) - \dot{z}^2e^{-2v(-y)}v'(-y) \\ 2\dot{y}\dot{z}e^{-2v(-y)}v'(-y) \\ -\dot{x}^2\dot{y}e^{2u(y)}(2u'^2(y) + u''(y)) - \dot{y}\dot{z}^2e^{-2v(-y)}(2v'^2(-y) - v''(-y)) \end{pmatrix}, \tag{35}$$

and here $L^1_1 = L_1$, $L^1_2 = L_2$, $L^1_3 = L_3$, $L^1_4 = L_4$. Iterate the previous calculation steps, and a first-order zero eigenvector λ^1 obtains

$$\lambda^1 = (2\dot{x}u'(y), 0, 2\dot{z}v'(-y), 1), \tag{36}$$

a secondary identity γ^1 then obtains

$$\gamma^1 = \lambda^1_{i_1}L^1_{i_1} = \lambda^1_{i_1}\alpha^1_{i_1} = -\dot{x}^2\dot{y}e^{2u(y)}u''(y) + \dot{y}\dot{z}^2e^{-2v(-y)}v''(-y) = 0, \tag{37}$$

whose total time differential is

$$\begin{aligned}\frac{d\gamma^1}{dt} &= -2\dot{x}\dot{y}e^{2u(y)}u''(-y)\ddot{x} - [\dot{x}^2e^{2u(y)}u''(y) - \dot{z}^2e^{-2v(-y)}v''(-y)]\ddot{y} + 2\dot{y}\dot{z}e^{-2v(-y)}v''(-y)\ddot{z} \\ &\quad - \dot{x}^2\dot{y}^2e^{2u(y)}[2u'(y)u''(y) + u'''(y)] + \dot{y}^2\dot{z}^2e^{-2v(-y)}[2v'(-y)v''(-y) + v'''(-y)] \\ &= 0.\end{aligned} \tag{38}$$

Take it as an addition L_5 to (33), which have to be taken by

$$L^2_{i_2} \equiv W^2_{i_2 j}\ddot{q}^j + \alpha^2_{i_2} = 0 \quad (i_2 = 1, 2, 3, 4, 5), \tag{39}$$

where

$$W^2 = \begin{pmatrix} e^{2u(y)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-2v(-y)} \\ -2\dot{x}e^{2u(y)}u'(y) & 0 & -2\dot{z}e^{-2v(-y)}v'(-y) \\ -2\dot{x}\dot{y}e^{2u(y)}u''(y) & [-\dot{x}^2e^{2u(y)}u''(y)+\dot{z}^2e^{-2v(-y)}v''(-y) & 2\dot{y}\dot{z}e^{-2v(-y)}v''(-y) \end{pmatrix}, \tag{40}$$

and

$$\alpha^2 = \begin{pmatrix} 2\dot{x}\dot{y}e^{2u(y)}u'(y) \\ -\dot{x}^2e^{2u(y)}u'(y) - \dot{z}^2e^{-2v(-y)}v'(-y) \\ 2\dot{y}\dot{z}e^{-2v(-y)}v'(-y) \\ -\dot{x}^2\dot{y}e^{2u(y)}[2u'^2(y)+u''(y)] - \dot{y}\dot{z}^2e^{-2v(-y)}[2v'^2(-y)-v''(-y)] \\ -\dot{x}^2\dot{y}^2e^{2u(y)}[2u'(y)u''(y)+u'''(y)] + \dot{y}^2\dot{z}^2e^{-2v(-y)}[2v'(-y)v''(-y)+v'''(-y)] \end{pmatrix}. \tag{41}$$

It is easy to see that a second-order zero eigenvector of (40) can be given as

$$\lambda^2 = (2\dot{x}u'(y), 0, 2\dot{z}v'(-y), 1, 0). \tag{42}$$

By left-multiplying λ^2 to (39), we obtain another secondary identity γ^2

$$\gamma^2 = \lambda^2_{i_3}L^2_{i_3} = \lambda^2_{i_3}\alpha^2_{i_3} = -\dot{x}^2\dot{y}e^{2u(y)}u''(y) + \dot{y}\dot{z}^2e^{-2v(-y)}v''(-y) = \gamma^1, \tag{43}$$

but which is not a new identity. Folding the previously discussed steps, we can not obtain any new identity. So the considered system has only two identities γ^0, and γ^1. Basing on (36) and (37), we can obtain

$$2\dot{x}u'(y)L_1 + 2\dot{z}v'(-y)L_3 + L_4 = 0. \tag{44}$$

Substitute (31) into (44), and it can be rewritten as

$$2\dot{x}u'(y)L_1 + \frac{d}{dt}L_2 + 2\dot{z}v'(-y)L_3 = 0 \tag{45}$$

with respect to (14). Refer to [21], and the relation between the gauge transformations and the gauge identities is

$$\delta q_i = \sum_{s=0}^{n}(-1)^s\frac{d^s w}{dt^s}\phi_{si}, \tag{46}$$

where ϕ_{si} can be deduced from (14). Following (46), the desired gauge transformations are

$$\begin{cases} \delta x = 2\dot{x}u'(y)w(t), \\ \delta y = \dot{w}(t), \\ \delta z = 2\dot{z}v'(-y)w(t), \end{cases} \tag{47}$$

where the dot denotes the time differential, and w denotes an arbitrary infinitesimal function depending time.

From (19), the time differential of coordinates x and z are

$$\dot{x} = p_x e^{-2u(y)}, \qquad \dot{z} = p_z e^{2v(-y)}. \tag{48}$$

As $w(t)$ satisfies the following relation

$$w(t) = -\dot{\varepsilon}_0(t), \tag{49}$$

its time differential has

$$\dot{w}(t) = -\ddot{\varepsilon}_0(t). \tag{50}$$

Substitute (48) into (47), and the required gauge transformations are

$$\begin{cases} \delta x = -2\dot{\varepsilon}_0(t)e^{-2u(y)}p_x u'(y), \\ \delta y = -\ddot{\varepsilon}_0(t), \\ \delta z = -2\varepsilon_0(t)e^{2v(-y)}p_z v'(-y). \end{cases} \tag{51}$$

Compare (51) with (23), and the difference of δy appears. If the δy in (51) equals the one in (47) strictly, there would be $-\ddot{\varepsilon}_0(t) = \varepsilon_0(t)$. $\varepsilon_0(t)$ is an arbitrary infinitesimal function depending time. Generally speaking, $\varepsilon_0(t)$ does satisfy the following inequality

$$-\ddot{\varepsilon}_0(t) \neq \varepsilon_0(t). \tag{52}$$

So the gauge transformations (51) are inequivalent to these (47). In other words, Dirac conjecture loses true to this dynamic system.

In what follows, for generality, another counterexample is introduced with a Lagrangian [22–27]

$$L = \dot{x}\dot{z} + xz - y\dot{z}. \tag{53}$$

Iterate the steps (16)–(23), the gauge transformations through Hamiltonian formalism are

$$\begin{cases} \delta x = \dot{\varepsilon}_0(t), \\ \delta y = \varepsilon_0(t), \\ \delta z = 0. \end{cases} \tag{54}$$

But repeat the steps (24)–(47), and the gauge transformations via Lagrangian formalism to this system are

$$\begin{cases} \delta x = w(t), \\ \delta y = 0, \\ \delta z = 0. \end{cases} \tag{55}$$

It is very clear that the gauge transformations (54) and (55) are different. Therefore, Dirac conjecture is invalid to this system, too.

4 An Example to Dirac Conjecture

It is well known that Dirac conjecture is effective to electromagnetic field. In what follows its gauge transformations are considered in Hamiltonian and Lagrangian formalism, respectively, too. A free electromagnetic field is the description of Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad (\mu, \nu = 0, 1, 2, 3), \tag{56}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and A_μ denotes the four-dimensional electromagnetic field.

In Legender transformation, canonical momentum π^μ conjugating to the canonical field A_μ can be given by

$$\pi^\mu = \frac{\partial L}{\partial \dot{A}_\mu} = -F^{0\mu}. \tag{57}$$

It is easy to see that there is only one primary constraint as

$$\pi^0(x) \approx 0. \tag{58}$$

With Dirac-Bergamman method, we obtain a secondary first-class constraint as

$$\partial_i \pi_i \approx 0. \tag{59}$$

In Dirac sense, Dirac conjecture is effective, so the generator of the desired gauge transformations is

$$G = \int [\varepsilon_0(t)\pi^0(x) + \dot{\varepsilon}_0(t)\partial_i \pi_i] dt, \tag{60}$$

and the gauge transformations are

$$\begin{cases} \delta A_0(x) = \varepsilon_0(t), \\ \delta A_i(x) = \dot{\varepsilon}_0(t)\partial_i \delta(x - x'). \end{cases} \tag{61}$$

In what follows the gauge transformations of the free electromagnetic field will be deduced in Lagrangian formalism again. In Lagrangian formalism, it is easy to calculate that this system satisfies the following differential equation

$$L^0_\mu = W^0_{\mu\nu} \ddot{A}_\nu(x) + \alpha^0_\mu \quad (\mu = 0, 1, 2, 3), \tag{62}$$

where

$$W^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{63}$$

and

$$\alpha^0 = \begin{pmatrix} \partial_i \dot{A}_0 + \partial_\mu F^{\mu 0} \\ -\ddot{A}_1 + \partial_\mu F^{\mu 1} \\ -\ddot{A}_2 + \partial_\mu F^{\mu 2} \\ -\ddot{A}_3 + \partial_\mu F^{\mu 3} \end{pmatrix}. \tag{64}$$

According to the matrix W^0, a zero-order zero eigenvector λ_0 can be easily given as

$$\lambda_0 = (1, 0, 0, 0), \tag{65}$$

which satisfies the following identity

$$\lambda_0 W^0 = 0. \tag{66}$$

In Lagrangian formalism, a primary identity of this system is

$$\gamma^0 = \lambda_0 \alpha^0 = \partial_i \dot{A}_0 + \partial_\mu F^{\mu 0} = 0, \tag{67}$$

and its time total differential is

$$\frac{d\gamma^0}{dt} = \partial_i \ddot{A}_0 + \partial_i^2 \dot{A}_0 - \partial_i \ddot{A}_i = 0. \tag{68}$$

According to the previous deduction, the motion equations of this system can be rewritten as

$$L_{i_1}^1 \equiv W_{i_1 j}^1 \ddot{q}^j + \alpha_{i_1}^1 = 0 \quad (j = 0, 1, 2, 3;\ i_1 = 0, 1, 2, 3, 4), \tag{69}$$

where

$$W^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \partial_i \delta(x - x') & -\partial_1 \delta(x - x') & -\partial_2 \delta(x - x') & -\partial_3 \delta(x - x') \end{pmatrix}, \tag{70}$$

and

$$\alpha_{i_1}^1 = \begin{pmatrix} \partial_i \dot{A}_0 \\ -\ddot{A}_1 \\ -\ddot{A}_2 \\ -\ddot{A}_3 \\ \partial_i^2 \dot{A}_0 \end{pmatrix}. \tag{71}$$

To matrix (70), a first-order zero eigenvector can be given by

$$\lambda_1 = (0, \partial_1 \delta(x - x'), \partial_2 \delta(x - x'), \partial_3 \delta(x - x'), 1). \tag{72}$$

Then a secondary identity can be easily obtained

$$\gamma^1 = \lambda_1 \alpha^1 = \frac{d}{dt}(\partial_i \pi^i) = 0, \tag{73}$$

and it has a total time differential form. So it can not bring out any new identity. The two identities (67) and (73) are pretty consistent with (58) and (59) in Hamiltonian formalism, so the required gauge transformations can be deduced [10, 28] as

$$\begin{cases} \delta A_0(x) = w_0(t), \\ \delta A_i(x) = \dot{w}_0(t) \partial_i \delta(x - x'). \end{cases} \tag{74}$$

It is easy to see that the gauge transformations (74) are completely consistent with these (61). The result keeps the validity of Dirac conjecture to free electromagnetic fields.

5 Conclusions

In this paper, how to solve the gauge identities for a constrained system has been reviewed first. The gauge identities in Lagrangian formalism are consistent with the first-class constraints in Hamiltonian formalism for a constrained dynamic system, in which Dirac conjecture is valid. Once the consistence vanishes to a system, in which Dirac conjecture would lose its validity. The gauge identities give rise to the gauge transformation via Lagrangian method, and the FCCs arise the gauge transformations through Hamiltonian approach. Therefore, the validity of Dirac conjecture can be told by whether the gauge transformations in Lagrangian formalism are equivalent to these in Hamiltonian formalism. This method is different from the old methods that they used expanded Hamiltonian, it depends on Lagrangian and total Hamiltonian of the constrained system. To prove this method, there are three examples, which include two counterexamples and an example to Dirac conjecture. The final results completely agree with the results discussed [29]. To constrained system with mixed constraints, whether the method is effective which needs to discuss again.

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